Lecture 31

Equivalence Theorems, Huygens' Principle

Electromagnetic equivalence theorems are useful for simplifying solutions to many problems. Also, they offer physical insight into the behaviour of electromagnetic fields of a Maxwellian system. They are closely related to Huygens' principle. One application is their use in studying the radiation from an aperture antenna or from the output of a lasing cavity. These theorems are discussed in many textbooks [33,51,55,66,196]. Some authors also call it Love's equivalence principles [197] and credit has been given to Schelkunoff as well [187].

You may have heard of another equivalence theorem in special relativity. It was postulated by Einstein to explain why light ray should bend around a star. The equivalence theorem in special relativity is very different from those in electromagnetics. One thing they have in common is that they are all derived by using Gedanken experiment (thought experiment), involving no math. But in this lecture, we will show, using mathematics, that electromagnetic equivalence theorems are also derivable albeit with more work.

31.1 Equivalence Theorems or Equivalence Principles

In this lecture, we will consider three equivalence theorems: (1) The inside out case. (2) The outside in case. (3) The general case. As mentioned above, we will derive these theorems using thought experiments or Gedanken experiments. As shall be shown later, they can also be derived mathematically using Green's theorem.

31.1.1 Inside-Out Case

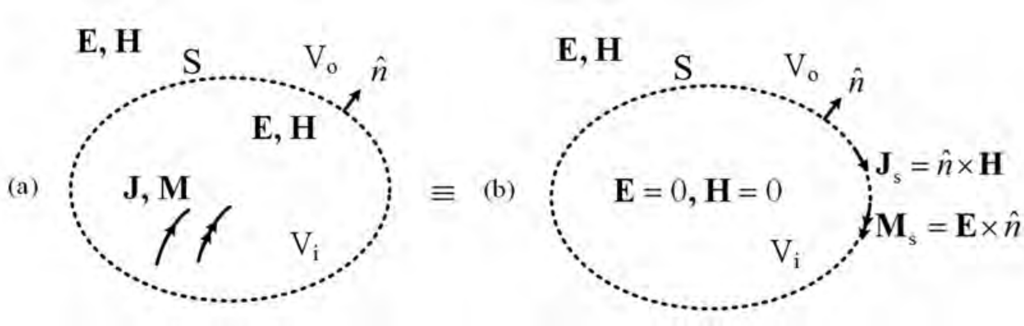

Figure 31.1: The inside-out problem where the two cases in (a) and (b) are equivalent. In (b), equivalence currents are impressed on the surface S so as to produce the same fields outside in V_o in both cases, cases (a) and (b).

In this case, as shown in Figure 31.1, we let $\mathbf{J}$ and $\mathbf{M}$ be the time-harmonic radiating sources inside a surface S radiating into a region $V = V_o \cup V_i$. They produce $\mathbf{E}$ and $\mathbf{H}$ everywhere. We call these fields and sources Maxwellian since they satisfy Maxwell's equations. We can construct an equivalence problem by first constructing an imaginary surface S. In this equivalence problem, the fields outside S in V_o are the same in both (a) and (b). But in (b), the fields inside S in V_i are zero. Despite, the fields and sources in (b) are Maxwellian.

Apparently, in case (b) in Figure 31.1, the tangential components of the fields are discontinuous at S. This is not possible for a Maxwellian fields unless surface currents are impressed on the surface S. We have learned from electromagnetic boundary conditions that electromagnetic fields are discontinuous across a current sheet. Then we ask ourselves what surface currents are needed on surface S so that the boundary conditions (or jump condition) for field discontinuities are satisfied on S. Clearly, surface currents needed for these field discontinuities. By virtue of the boundary conditions and the jump conditions in electromagnetics, these surface currents to be impressed on S are

$$\mathbf{J}_s = \hat{n} \times \mathbf{H}, \qquad \mathbf{M}_s = \mathbf{E} \times \hat{n} \tag{31.1.1}$$

We have learnt from Section 4.3.3 that an electric current sheet in Ampere's law produced a jump discontinuity in the magnetic field. By the same token, fictitious magnetic current is added to Faraday's law in Section 5.3 for mathematical symmetry. Then by duality, a magnetic current sheet induces a jump discontinuity in the electric field. Because of the opposite polarity of the magnetic current $\mathbf{M}$ in Faraday's law compared to the electric current in Ampere's law as is shown in Section 5.3, this magnetic current sheet is proportional to $\mathbf{E} \times \hat{n}$ instead of $\hat{n} \times \mathbf{H}$.

Consequently, we can convince ourselves that $\hat{n} \times \mathbf{H}$ and $\mathbf{E} \times \hat{n}$ just outside S in both cases are the same. Furthermore, we are persuaded that the above is a bona fide solution to Maxwell's equations. This fact can be proved, as shall be shown later on, by a more mathematical manipulation. The fact that the field is zero in V_i is known as the *extinction theorem.*

The fact that these equivalence currents generate zero fields inside S is known as the *extinction theorem.* This equivalence theorem can also be proved mathematically, as shall be shown.

31.1.2 Outside-in Case

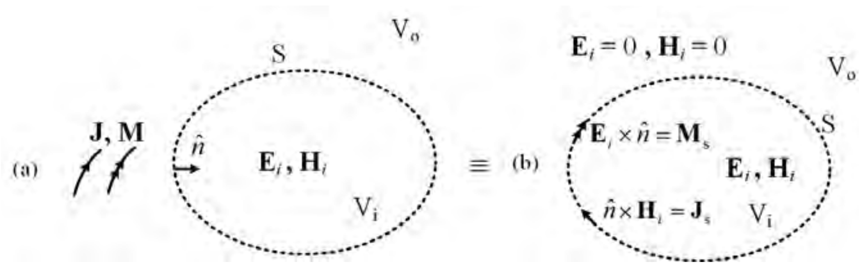

Figure 31.2: The outside-in problem where equivalence currents are impressed on the surface S to produce the same fields inside in both cases.

Similar to before, in Figure 31.2, we find an equivalence problem (b) where the fields inside S in V_i is the same as in (a), but the fields outside S in V_o in the equivalence problem is zero. The fields are discontinuous across the surface S, and hence, impressed surface currents are needed to account for these discontinuities.

Then by the uniqueness theorem,[1] the fields $\mathbf{E}_i$, $\mathbf{H}_i$ inside V in both cases are the same. Again, by the *extinction theorem*, the fields produced by $\mathbf{E}_i \times \hat{n}$ and $\hat{n} \times \mathbf{H}_i$ are zero outside S.

It is to be noted that for both inside-out and outside-in cases, the field is extinct by the extinction theorem only in the volume or region that originally contains the sources. This will be clear when these equivalence problems are derived mathematically.

31.1.3 General Case

From these two cases, we can create a rich variety of equivalence problems. By linear superposition of the inside-out problem, and the outside-in problem, then a third equivalence problem is shown in Figure 31.3:

[1]We can add infinitesimal loss to ensure that uniqueness theorem is satisfied in this enclosed volume V_i.

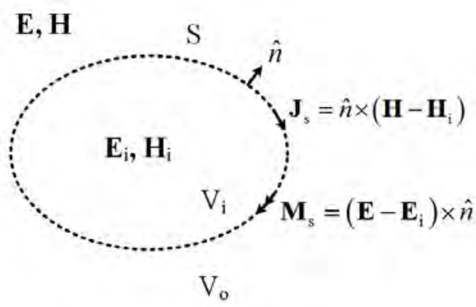

Figure 31.3: The general case where the fields are non-zero both inside and outside the surface S. Equivalence currents are needed on the surface S to support the jump discontinuities in the fields.

31.2 Electric Current on a PEC

Using the equivalence problems in the previous section, we can derive other corollaries of equivalence theorems. We shall show them next.

First, from reciprocity theorem, it is quite easy to prove that an impressed current on the PEC cannot radiate. We can start with the inside-out equivalence problem as shown in (b) of Figure 31.1. Since the fields inside S is zero for the inside-out problem, using a Gedanken experiment , one can insert an PEC object inside S without disturbing the fields $\mathbf{E}$ and $\mathbf{H}$ outside since the field is zero inside S. As the PEC object grows to snugly fit the surface S, then the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$, which is an impressed current source on top of a PEC, does not radiate by reciprocity. Only one of the two currents is radiating, namely, the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ is radiating; and hence, $\mathbf{J}_s$ in Figure 31.4 can be removed. This is commensurate with the uniqueness theorem that only the knowledge of $\mathbf{E} \times \hat{n}$ is needed to uniquely determine the fields outside S.

It is to be noted that $\mathbf{J}_s$, $\mathbf{M}_s$, $\mathbf{E}$ and $\mathbf{H}$ form a Maxwellian system before we insert a PEC object inside the surface S shown in (b) in Figure 31.1.

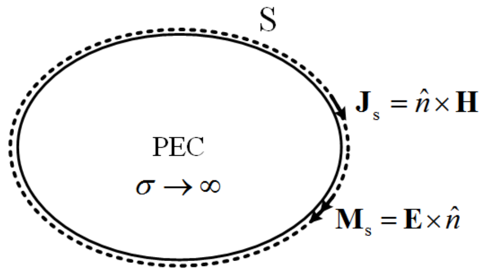

Figure 31.4: On a PEC surface, only one of the two currents is needed since an electric current does not radiate on a PEC surface.

31.3 Magnetic Current on a PMC

Again, from reciprocity, an impressed magnetic current on a PMC cannot radiate. By the same token, we can perform the Gedanken experiment as before by inserting a PMC object inside S. It will not alter the fields outside S, as the fields inside S is zero. As the PMC object grows to snugly fit the surface S, only the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ radiates, and the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ does not radiate and it can be removed. This is again commensurate with the uniqueness theorem that only the knowledge of the $\hat{n} \times \mathbf{H}$ is needed to uniquely determine the fields outside S.

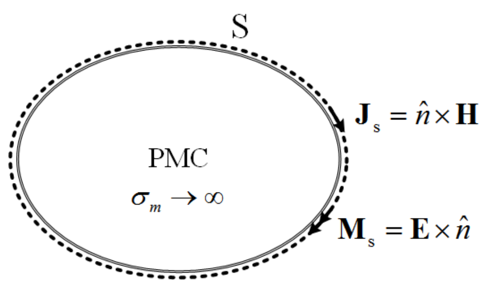

Figure 31.5: Similarly, on a PMC surface, only an electric current is needed to produce the field outside the surface S.

31.4 Huygens' Principle and Green's Theorem

Huygens' principle shows how a wave field on a surface determines the wave field outside the surface S. This concept was expressed by Huygens heuristically in the 1600s [198]. But the mathematical expression of this idea was due to George Green[2] in the 1800s. This concept

[2]George Green (1793-1841) was self educated and the son of a miller in Nottingham, England [199].

can be expressed mathematically for both scalar and vector waves. The derivation for the vector wave case is homomorphic to the scalar wave case. But the algebra in the scalar wave case is much simpler. Therefore, we shall discuss the scalar wave case first, followed by the electromagnetic vector wave case.

31.4.1 Scalar Waves Case

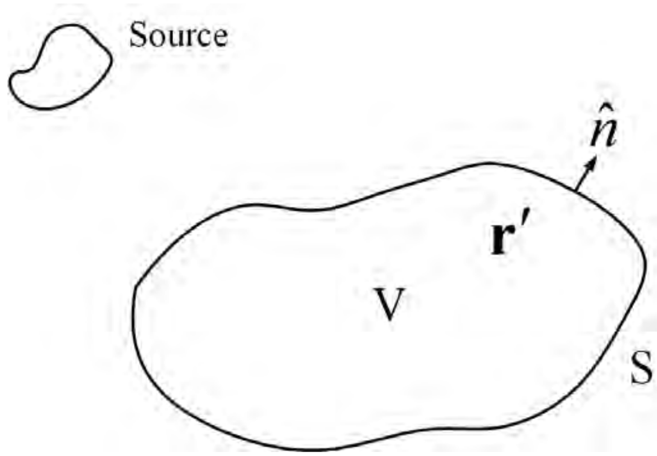

Figure 31.6: The geometry for deriving Huygens' principle for scalar wave equation.

For a $\psi(\mathbf{r})$ that satisfies the scalar wave equation inside V

$$(\nabla^2 + k^2)\,\psi(\mathbf{r}) = 0, \tag{31.4.1}$$

Notice that V does not contain the source that produces $\psi(\mathbf{r})$ so that the right-hand side of the above can be set to zero always. The corresponding scalar Green's function $g(\mathbf{r}, \mathbf{r}')$ satisfies

$$(\nabla^2 + k^2)\,g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{31.4.2}$$

Next, we multiply (31.4.1) by $g(\mathbf{r}, \mathbf{r}')$ and (31.4.2) by $\psi(\mathbf{r})$. And then, we subtract the two resultant equations and integrating over a volume V as shown in Figure 31.6. There are two cases to consider: when $\mathbf{r}'$ is in V, or when $\mathbf{r}'$ is outside V. Thus, we have

$$\left.\begin{array}{ll} \text{if } \mathbf{r}' \in V, & \psi(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array}\right\} = \int_V d\mathbf{r}\,[g(\mathbf{r}, \mathbf{r}')\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2 g(\mathbf{r}, \mathbf{r}')], \tag{31.4.3}$$

The left-hand side evaluates to different values depending on where $\mathbf{r}'$ is due to the sifting property of the delta function $\delta(\mathbf{r}-\mathbf{r}')$. Since $g\nabla^2\psi - \psi\nabla^2 g = \nabla\cdot(g\nabla\psi - \psi\nabla g)$, the right-hand side of (31.4.3) can be rewritten using Gauss' divergence theorem, giving[3]

$$\left.\begin{array}{ll}\text{if } \mathbf{r}' \in V, & \psi(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0\end{array}\right\} = \oint_S dS\, \hat{n} \cdot [g(\mathbf{r}, \mathbf{r}')\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla g(\mathbf{r}, \mathbf{r}')], \tag{31.4.4}$$

where S is the surface bounding V. The above is the Green's theorem, or the mathematical expression that once $\psi(\mathbf{r})$ and $\hat{n} \cdot \nabla\psi(\mathbf{r})$ are known on S, then $\psi(\mathbf{r}')$ away from S could be found. This is similar to the expression of equivalence principle where $\hat{n} \cdot \nabla\psi(\mathbf{r})$ and $\psi(\mathbf{r})$ are equivalence sources on the surface S.

The first term on the right-hand side radiates via the Green's function $g(\mathbf{r}, \mathbf{r}')$. Since this is a monopole field, this source is also called a monolayer or single layer source. The second term radiates, on the other hand, via the normal derivative of the Green's function, namely $\hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}')$. Since the derivative of a Green's function yields a dipole field, the second term corresponds to sources that radiate like dipoles pointing normally to the surface S. These sources are also called double layer (or dipole layer) sources. These terminologies are prevalent in acoustics. The above mathematical expression also embodies the *extinction theorem* that says if $\mathbf{r}'$ is outside V, the left-hand side evaluates to zero.

It is to be noted that the left-hand side of (31.4.4) is the mathematical expression of the extinction theorem. It would not have been possible, if the right-hand side of (31.4.1) is not zero. In other words, we have picked V so that the right-hand side (31.4.1) can be set to zero. Thus the field is always extinct in the volume or region that contains the source (you may need to think about this a bit:)).

[3]The equivalence of the volume integral in (31.4.3) to the surface integral in (31.4.4) is also known as Green's theorem [92].

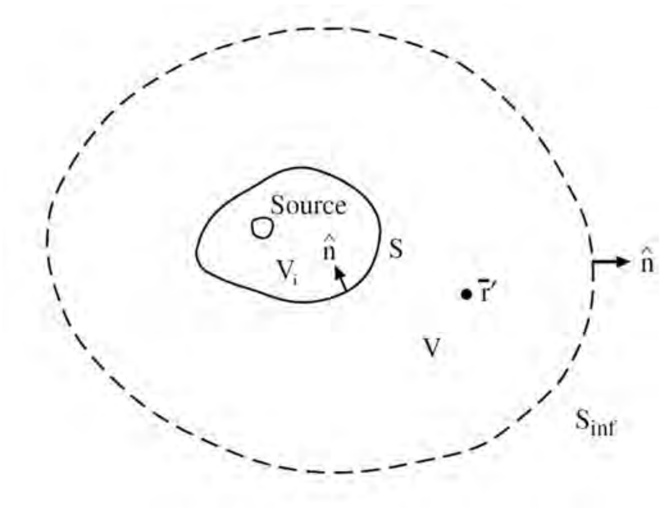

Figure 31.7: The geometry for deriving Huygens' principle for scalar wave. The radiation from the source can be replaced by equivalence sources on the surface S, and the field outside S can be calculated using (31.4.4). Also, the field is zero inside S from (31.4.4). This is the extinction theorem.

If the volume V is bounded by S and S_{inf} as shown in Figure 31.7, then the surface integral in (31.4.4) should include an integral over S_{inf}. But when $S_{\text{inf}} \to \infty$, all fields look like plane wave, and $\nabla \to -\hat{r}jk$ on S_{inf}. Furthermore, $g(\mathbf{r}-\mathbf{r}') \sim O(1/r)$, when $r \to \infty$, and $\psi(\mathbf{r}) \sim O(1/r)$, when $r \to \infty$, if $\psi(\mathbf{r})$ is due to a source of finite extent.[4] Then, the integral over S_{inf} in (31.4.4) vanishes, and (31.4.4) is valid for the case shown in Figure 31.7 as well but with the surface integral over surface S only.

Here, the field outside S at $\mathbf{r}'$ is expressible in terms of the field on S. This is similar to the inside-out equivalence principle we have discussed previously Section 31.1.1, albeit this is for scalar wave case.

Notice that in deriving (31.4.4), $g(\mathbf{r},\mathbf{r}')$ has only to satisfy (31.4.2) for both $\mathbf{r}$ and $\mathbf{r}'$ in V but no boundary condition has yet been imposed on $g(\mathbf{r},\mathbf{r}')$. Therefore, if we further require that $g(\mathbf{r},\mathbf{r}')=0$ for $\mathbf{r}\in S$, then (31.4.4) becomes

$$\psi(\mathbf{r}') = -\oint_S dS\,\psi(\mathbf{r})\,\hat{n}\cdot\nabla g(\mathbf{r},\mathbf{r}'), \qquad \mathbf{r}'\in V. \tag{31.4.5}$$

On the other hand, if require additionally that $g(\mathbf{r},\mathbf{r}')$ satisfies (31.4.2) with the boundary condition $\hat{n}\cdot\nabla g(\mathbf{r},\mathbf{r}')=0$ for $\mathbf{r}\in S$, then (31.4.4) becomes

$$\psi(\mathbf{r}') = \oint_S dS\,g(\mathbf{r},\mathbf{r}')\,\hat{n}\cdot\nabla\psi(\mathbf{r}), \qquad \mathbf{r}'\in V. \tag{31.4.6}$$

[4]The symbol "O" means "of the order."

Equations (31.4.4), (31.4.5), and (31.4.6) are various forms of Huygens' principle, or equivalence principle for scalar waves (acoustic waves) depending on the definition of $g(\mathbf{r}, \mathbf{r}')$. Equations (31.4.5) and (31.4.6) stipulate that only $\psi(\mathbf{r})$ or $\hat{n} \cdot \nabla\psi(\mathbf{r})$ need be known on the surface S in order to determine $\psi(\mathbf{r}')$. The above are analogous to the PEC and PMC equivalence principle considered previously in Sections 31.2 and 31.3.

31.4.2 Electromagnetic Waves Case

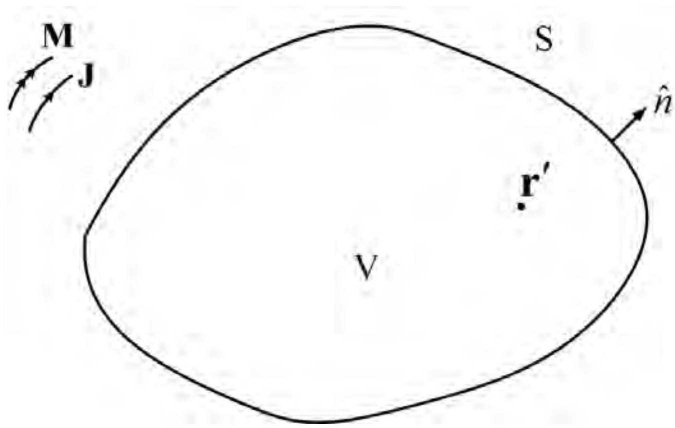

Figure 31.8: Derivation of the Huygens' principle for the electromagnetic case. One only needs to know the surface fields on surface S in order to determine the field at $\mathbf{r}'$ inside V.

The derivation of Huygens' principle and Green's theorem for the electromagnetic case is more complicated than the scalar wave case. But fortunately, this problem is mathematically homomorphic to the scalar wave case. In dealing with the requisite vector algebra, we have to remember to cross the t's and dot the i's, to carry ourselves carefully through the laborious and complicated vector algebra! One can always refer back to the scalar-wave case to keep our bearing straight.

In a source-free homogeneous region, an electromagnetic wave satisfies the vector wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\,\mathbf{E}(\mathbf{r}) = 0. \tag{31.4.7}$$

Again, we pick the volume V so that the right-hand side of the above is zero to simplify the derivations. The analogue of the scalar Green's function for the scalar wave equation is the dyadic Green's function for the electromagnetic wave case [1, 33, 200, 201]. Moreover, the dyadic Green's function satisfies the equation[5]

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2 \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}} \, \delta(\mathbf{r} - \mathbf{r}'). \tag{31.4.8}$$

It can be shown by direct back substitution that the dyadic Green's function in free space is [201]

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\bar{\mathbf{I}} + \frac{\nabla \nabla}{k^2} \right) g(\mathbf{r} - \mathbf{r}') \tag{31.4.9}$$

The above allows us to derive the vector Green's theorem [1, 33, 200].

Then, after post-multiplying (31.4.7) by $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$, pre-multiplying[6] (31.4.8) by $\mathbf{E}(\mathbf{r})$, subtracting the resultant equations, the terms involving $k^2 \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ cancel out. We then integrate the difference over volume V, and using the sifting property of the delta function, considering two cases as we did for the scalar wave case, we have

$$\left. \begin{array}{ll} \text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array} \right\} = \int_V dV \, \left[\mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right. \\ \left. - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right]. \tag{31.4.10}$$

Next, using the vector identity that[7]

$$-\nabla \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right] \\ = \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \tag{31.4.11}$$

then the integrand of (31.4.10) can be written as a divergence. With the help of Gauss' divergence theorem, the right-hand side of (31.4.10) can be written as

$$\left. \begin{array}{ll} \text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array} \right\} = - \oint_S dS \, \hat{n} \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right] \\ = - \oint_S dS \, \left[-\mathbf{E}(\mathbf{r}) \times \hat{n} \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + i\omega\mu \, \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right]. \tag{31.4.12}$$

Again, the left-hand side of the above is simple and embodies the extinction theorem. It would not have been that simple if the right-hand side of (31.4.7) has not been made zero with a proper choice of V. Hence, the field is extinct in the volume that contains the sources.

The above is just the vector analogue of (31.4.4). We have used the cyclic relation of dot and cross products to rewrite the last expression. Since $\hat{n} \times \mathbf{E}$ and $\hat{n} \times \mathbf{H}$ are associated

[5]A dyad is an outer product between two vectors, and it behaves like a tensor, except that a tensor is more general than a dyad. A purist will call the above a tensor Green's function, as the above is not a dyad in its strictest definition.

[6]Since we are dealing with dyads which are tensors like matrices, order is very important here.

[7]This identity can be established by using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$. We will have to let (31.4.11) act on an aribitrary constant vector to convert the dyad into a vector before applying this identity. The equality of the volume integral in (31.4.10) to the surface integral in (31.4.12) is also known as the vector Green's theorem [33, 200]. Earlier form of this theorem was known as Franz formula [202].

with surface magnetic current $\mathbf{M}_s$ and surface electric current $\mathbf{J}_s$, respectively, the above can be thought of having these equivalence surface currents radiating via the dyadic Green's function. Again, notice that (31.4.12) is derived via the use of (31.4.8), but no boundary condition has yet been imposed on $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ on S even though we have given a closed form solution for the free-space case. The above is similar to the outside-in equivalence theorem we have derived in Section 31.1.2 using a Gedanken experiment . Now we have a mathematical derivation of the same theorem!

Now, if we require the additional boundary condition that $\hat{n} \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$. This corresponds to a point source located at $\mathbf{r}'$ radiating via the dyadic Green's function, producing a field at $\mathbf{r}$, in the presence of a PEC surface. Then (31.4.12) becomes

$$\mathbf{E}(\mathbf{r}') = -\oint_S dS\, \hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V \tag{31.4.13}$$

for it could be shown that $\hat{n} \times \mathbf{H} \cdot \overline{\mathbf{G}} = \mathbf{H} \cdot \hat{n} \times \overline{\mathbf{G}}$ implying that the second term in (31.4.12) is zero. On the other hand, if we require that $\hat{n} \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (31.4.12) becomes

$$\mathbf{E}(\mathbf{r}') = -i\omega\mu \oint_S dS\, \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V \tag{31.4.14}$$

Equations (31.4.13) and (31.4.14) state that $\mathbf{E}(\mathbf{r}')$ is determined if either $\hat{n} \times \mathbf{E}(\mathbf{r})$ or $\hat{n} \times \mathbf{H}(\mathbf{r})$ is specified on S. This is in agreement with the uniqueness theorem. These are the mathematical expressions of the PEC and PMC equivalence problems we have considered previously in Sections 31.2 and 31.3.

The dyadic Green's functions in (31.4.13) and (31.4.14) are for a closed cavity since boundary conditions are imposed on S for them. But the dyadic Green's function for an unbounded, homogeneous medium, given in (31.4.10) can be written as

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2} [\nabla \times \nabla \times \bar{\mathbf{I}}\, g(\mathbf{r} - \mathbf{r}') - \bar{\mathbf{I}}\, \delta(\mathbf{r} - \mathbf{r}')], \tag{31.4.15}$$

$$\nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \nabla \times \bar{\mathbf{I}}\, g(\mathbf{r} - \mathbf{r}'). \tag{31.4.16}$$

Then, (31.4.12), for $\mathbf{r}' \in V$ and $\mathbf{r}' \neq \mathbf{r}$, becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla' \times \oint_S dS\, g(\mathbf{r} - \mathbf{r}')\, \hat{n} \times \mathbf{E}(\mathbf{r}) + \frac{1}{i\omega\epsilon} \nabla' \times \nabla' \times \oint_S dS\, g(\mathbf{r} - \mathbf{r}')\, \hat{n} \times \mathbf{H}(\mathbf{r}). \tag{31.4.17}$$

The above can be applied to the geometry in Figure 31.7 where $\mathbf{r}'$ is enclosed in S and S_{inf}. However, the integral over S_{inf} vanishes by virtue of the radiation condition as for (31.4.4).[8] Then, (31.4.17) relates the field outside S at $\mathbf{r}'$ in terms of only the equivalence surface currents $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ and $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ on S. This is similar to the inside-out problem in the equivalence theorem (see Section 31.1.1). It is also related to the fact that if the radiation condition is satisfied, then the field outside of the source region is uniquely satisfied. Hence, this is also related to the uniqueness theorem.

[8]It is to be noted that the integral over S_{inf} does not vanish because the field is vanishingly small, but the cancellation of the two terms in (31.4.17).